LINEAR IDENTIFICATION OF LINEAR RATIONAL-EXPECTATIONS MODELS WITH EXOGENOUS VARIABLES

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* **This work represents entirely the author's own work.**

SUMMARY OF PAPER

- Leading (expected future) effects are as important as lagged effects for determining expected economic paths.

- Commonly used VAR models explicitly include only lagged effects, maybe only indirectly via survey variables.

- LREMs with exogenous variables are VARX models with added expected-future terms in (usually) endogenous and (sometimes) exogenous variables.

- RE modelling has been the only systematic L & NL method for modelling both lagged and leading effects.

- RE models (now mostly called DSGEMs) have been largely motivated by economic theory, which has disadvantages:

  1. Need more demanding MLE or Bayesian NL estimation in terms of NL restricted "deep" parameters.

2. Have zero restrictions on coefficients that can put models in unsatisfactory "corners" of model spaces.

3. If necessary (poor fit or predictions), have narrow room for modification except with more theorizing.

- Paper derives indirect linear least-squares estimates (ILLSE) of structural and "RES" coefficients of a LREM, without any restrictions on coefficients, only one standard normalization of disturbance covariances.

- Resulting "purely data based" ILLSE model can be compared for in-sample fit or out-of-sample prediction with a commonly estimated LREM with NL restrictions and modified in its direction or vice versa.

- ILLSE RES equation can be used to make Lucas-consistent econometric policy evaluations.

- Basically, LREM I/E extends LSEM I/E; e.g., slopes of D&S curves shifted by different exogenous variables.

ORGANIZATION

1. INTRODUCTION.
2. IDENTIFICATION 1.
3. IDENTIFICATION 2.
4. CONCLUSIONS.
5. REFERENCES.

1. INTRODUCTION.

[1] Linear rational expectation models (LREMs):

- LREMs are standard multivariate structural macroeconomic models that extend linear simultaneous equations models (LSEMs) by adding terms in expected-future variables.

- LREMs are now commonly called dynamic stochastic general equilibrium models (DSGEMs) to emphasize generality and nonlinearity in variables; here we call them LREMs because we only consider models linear in variables.

- Goal of paper: extend identification of LSEMs with exogenous variables to LREMs.

- "With" means (i) a model has exogenous variables and (ii) they are the key to identification without using any theoretical restrictions except one matrix normalization, so that the identification is "purely data based".

[2] LREM has 4 equations:

(1) Structural/endogenous: $A_2E_ty_{t+1}+A_1y_t+A_0y_{t-1} = B_0\varepsilon_t+C_0z_t$,

(2) RES/endogenous: $y_t = \Phi_1y_{t-1} + \Theta_0\varepsilon_t + \sum_{i=0}^{\infty}\Xi_iE_tz_{t+i}$,

(3) Structural/RF/exogenous: $z_t = D_1z_{t-1} + D_2z_{t-2} + \xi_t$,

(4) RF/endogenous: $y_t = \Phi_1y_{t-1} + \Theta_0\varepsilon_t + \Upsilon_0z_t + \Upsilon_1z_{t-1}$,

y_t = n×1 observed endogenous variables,
ε_t = n×1 unobserved disturbances ~ $IID(0_{nx1},I_n)$,
z_t = m×1 observed exogenous variables, $m \geq n$,
ξ_t = m×1 unobserved disturbances ~ $IID(0_{nx1},\Sigma_\xi)$.

- Eqs. (1) & (3) are specified.
- Solving eq. (1) => eq. (2).
- Combining eqs. (2) & (3) => (4).
- Combining eqs. (3) & (4) => RF VAR of all variables.

[3] 4 forward-solution functions from bottom-to-top:

- 5 levels of quantities:

1. θ_1=vector of "deep structural parameters".
2. $\theta_2=\{\{A_2\}_{i=0}^{2}, B_0, C_0\}$= matrices of "structural coefficients".
3. $\theta_3=\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$= matrices of "RES coefficients".
4. $\theta_4=\{\Phi_1, \Theta_0, \{\Upsilon_i\}_{i=0}^{1}, \{D_i\}_{i=1}^{2}\}$= matrices of "RF coefficients".
5. θ_5=vector of "data moments".

- 4 forward-solution functions,

θ_1 => $\theta_2=f_1(\theta_1)$ => $\theta_3=f_2(\theta_2)$ => $\theta_4=f_3(\theta_3)$ => $\theta_5=f_4(\theta_4)$, or

$\theta_{i+1} = f_i(\theta_i)$, $i = 1,\ldots,4$.

- 10 composite forward-solution functions,

$\theta_j = g_{ji}(\theta_i) \triangleq f_{j-1}(\cdots f_i(\theta_i)\cdots)$, for $1,\cdots,4 = i < j = 2,\cdots,5$.

[4] 4 inverse-identification functions from top-to-bottom:

- 4 inverse-identification functions,

$\theta_4 = f_4^{-1}(\theta_5) \Rightarrow \theta_3 = f_3^{-1}(\theta_4) \Rightarrow \theta_2 = f_2^{-1}(\theta_3) \Rightarrow \theta_1 = f_1^{-1}(\theta_2)$, or

$\theta_{i-1} = f_{i-1}^{-1}(\theta_i)$, $i = 5,\ldots,2$.

- 10 composite inverse-identification" functions:

For $1,\cdots,4 = i < j = 2,\cdots,5$:

$\theta_i = g_{ij}^{-1}(\theta_j) \triangleq f_i^{-1}(\cdots f_{j-1}^{-1}(\theta_j)\ldots)$;

$g_{ij}^{-1}(\theta_j)$ exists because $\theta_j = g_{ji}(\theta_i)$;

θ_i is identified from θ_j if & only if $g_{ij}^{-1}(\theta_j)$ is unique.

- Paper considers only:

Identification 1: $\theta_2 = f_2^{-1}(\theta_3)$ of structural from RES coeffs.

Identification 2: $\theta_3 = f_3^{-1}(\theta_4)$ of RES from RF coefficients.

- Because the goal is to obtain general results and $\theta_1 = f_1^{-1}(\theta_2)$ depends on a particular LREM, it's not considered.

- Because $\theta_4 = f_4^{-1}(\theta_5)$ just says that linear least-squares estimation is consistent, it's also not considered.

- Combined identification 1-2 of $\theta_2 = g_{24}^{-1}(\theta_4)$ identifies structural coefficients from RF coefficients via RES coefficients.

- Goal of paper: derive equations for evaluating $\theta_2 = f_2^{-1}(\theta_3)$ and $\theta_3 = f_3^{-1}(\theta_4)$ and prove the inverse functions are unique under assumptions (A.1)-(A.6) on structural coefficients.

[5] Linear versus nonlinear identification:

- General discussion of LREM identification started about 10 years ago: Iskrev (2009), Komunjer & Ng (2011), Qu & Tkachenko (2012, 2017), Kociecki & Kolasa (2018, 2020), Al-Sadoon & Zwiernik (2020), ...

- All emphasized nonlinearity of LREM identification.

- However, only identifications $\theta_1 = f_1^{-1}(\theta_2)$ and $\theta_4 = f_4^{-1}(\theta_5)$ might be nonlinear, most likely only $\theta_1 = f_1^{-1}(\theta_2)$.

- Identification $\theta_4 = f_4^{-1}(\theta_5)$ might be nonlinear only due to data complications such as aliasing (Hansen & Sargent, 1982) due to mixed frequencies or subsampling (Anderson et al., 2012; Zadrozny, 2016; Tank et al., 2019).

- **Present identifications 1-2 are linear.**

[6] Practical implications of identifications 1-2:

- **Identification 1 as estimation (down from consistently estimated data moments) produces purely-data-based (without theoretical restrictions) estimated structural coefficients, so that it gives a purely-data-based idea of the coefficients' signs and distances from zero.**

- **Identification 2 as estimation produces purely-data-based RES equations that can predict consistently with Lucas's (1976) critique of econometric policy evaluation.**

- **RES-equation predictions are consistent with Lucas's critique because RES coefficients $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$ are independent of paths of current and expected-future exogenous variables $\{E_t z_{t+i}\}_{i=0}^{\infty}$.**

2. IDENTIFICATION 1.

[1] Cross equations of rational expectations (CERRE):

- In forward solution, for given structural coefficients $\{\{A_2\}_{i=0}^{2}, B_0, C_0\}$, RES coefficients $\{\Phi_1, \Theta_0, \{\Xi_i\}_{i=0}^{\infty}\}$ satisfy

(5) $$A_2\Phi_1^2 + A_1\Phi_1 + A_0 = 0_{nxn},$$

(6) $$(A_2\Phi_1 + A_1)\Theta_0 = B_0,$$

(7) $$(A_2\Phi_1 + A_1)\Xi_0 = C_0,$$

(8) $$(A_2\Phi_1 + A_1)\Xi_i + A_2\Xi_{i-1} = 0_{nxn}, \text{ for } i = 1,2,3,\ldots,$$

and inversely in identification.

- Φ_1^2 in eq. (5) makes forward solution with the CERRE nonlinear, but any inverse identification with them is linear.

[2] One version of identification 1 is ...

- If $n = \#(y_t) = \#(z_t)$ and $C_0 = I_n$, then, CERRE (7)-(8) imply

(9) $$XM = N, \quad X = [A_2, A_1],$$

$$M = \left\{ \begin{bmatrix} I_n & \Phi_1 \\ 0_{nxn} & I_n \end{bmatrix} \begin{bmatrix} I_n & 0_{nxn} \\ \Xi_1 \Xi_0^{-1} & I_n \end{bmatrix} P \begin{bmatrix} \Xi_0 & 0_{nxn} \\ 0_{nxn} & \Xi_0 \end{bmatrix} \right\},$$

$$N = [I_n, 0_{nxn}], \quad P = \text{row-permuted } I_n.$$

- Because RES coefficients $\{\Phi_1, \{\Xi_i\}_{i=0}^{1}\}$ are known, M is known; because Ξ_0 is nonsingular, M is nonsingular.

- Therefore, structural coefficients $\{A_2\}_{i=1}^{2}$ are identified by $X = NM^{-1}$, whereupon remaining structural coefficients A_0 and B_0 are identified by eqs. (5)-(6).

3. IDENTIFICATION 2.

[1] One version of identification 2 is ...

- If $n = \#(y_t) = \#(z_t)$ and $C_0 = I_n$, then, eqs. (2)-(3) imply

$$(10)\quad XM = N,\ X = [\Pi, \Xi_0],\ M = \begin{bmatrix} \Upsilon_0 D_1 + \Upsilon_1 & \Upsilon_0 D_2 \\ I_n & 0_{nxn} \end{bmatrix},\ N = [\Upsilon_0, \Upsilon_1],$$

and CERRE (8) is restated aS

$$(11)\quad \Xi_i = \Pi \Xi_{i-1},\ \text{for } i = 1,2,3,\dots,$$

- Because RF coefficients $\{\{\Upsilon_i\}_{i=0}^{1}, \{D_i\}_{i=1}^{2}\}$ are known, $\{M, N\}$ are known; because $A_2\Phi_1 + A_1$ is nonsingular, $\Pi = -(A_2\Phi_1 + A_1)^{-1} A_2$ exists; because $\Upsilon_0 D_2$ is nonsingular, M is nonsingular; all nonsingularities are proved in the paper.

- Therefore, $\{\Pi, \Xi_0\}$ are identified by $X = NM^{-1}$, whereupon $\{\Xi_i\}_{i=1}^{K}$ are identified by iterating on eq. (11).

4. CONCLUSIONS.

[1] Extensions:

(i) Add z_{it}'s, so that $\#(z_t) = m > n = \#(y_t)$.

(ii) Add lags of z_t.

(iii) Add expected leads of z_t.

(iv) Add expected leads and lags of y_t.

(v) Add lags of ε_t and ξ_t.

- The paper considers extensions (i)-(iii); all of the extensions follow the same logic as above, but result in different linear-identification eqs. $XM = N$, with different details and conditions for row rank(M) = full, so that identification $X = NM^T(MM^T)^{-1}$ holds.

[2] Practical implications of identifications 1-2.

- Identification 2 as consistent estimation (when assumed data moments are replaced by consistently estimated data moments) produces a linearly purely-data-based estimated LREM (without theoretical restrictions on coefficients).

- A RES equation of the linearly estimated LREM can make predictions that are consistent with Lucas's (1976) critique of econometric policy evaluation.

- A linearly estimated LREM can be compared in terms of in-sample fit and out-of-sample predictions with a commonly nonlinearly estimated LREM in terms of nonlinearly restricted deep parameters.

- A linearly estimated LREM can be a starting point or benchmark for developing a commonly nonlinearly estimated LREM, which could thereby fit data and predict better than an commonly nonlinearly estimated LREM motivated only or more narrowly by theoretical considerations.

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