Rational Inattention in the Frequency Domain

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Overview

This paper solves the canonical rational inattention (RI) tracking problem by formulating it in the frequency domain.

- **Main result**: RI version of the Wiener-Kolmogorov filtering formulas
  - Don't require the target process to have a state-space representation
  - Shed new light on why RI produces forward-looking behavior
  - Facilitate acquisition of closed-form solutions in certain cases
  - Can be implemented numerically by a simple algorithm (thanks to FFT)

Problem

Track a stationary target process $x_t = \sum_{s=-\infty}^{\infty} a_s \varepsilon_{t-s}$, $\varepsilon_t \sim N(0, I)$ by choosing an action process y to solve

$$\inf \; E[(x_t - y_t)'(x_t - y_t)] \quad \text{s.t.}$$

$$\lim_{T\to\infty} \frac{1}{T} I((\varepsilon_{t+1}, \ldots, \varepsilon_{t+T}), (y_{t+1}, \ldots, y_{t+T})) \leq \kappa \qquad \text{(processing)}$$

$$I((\varepsilon_{t+\tau+1}, \varepsilon_{t+\tau+2} \ldots), y^t | \varepsilon^{t+\tau}) = 0 \qquad \text{(availability)}$$

Proposition 1. *There exists a solution, and the distribution of the optimal pair* (ε, y) *is unique and Gaussian.*

Frequency domain

By Gaussianity: $y_t = \sum_{s=-\infty}^{\infty} b_s \varepsilon_{t-s} + v_t$, $g_s = E v_t v_{t-s}'$. Turning sequences into functions, e.g. $a(\lambda) \equiv \sum_{s=-\infty}^{\infty} a_s e^{-i\lambda s}$, the problem becomes

$$\min_{b, g \geq 0} \; \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}[(a-b)(a-b)^* + g] d\lambda \quad \text{s.t.}$$

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \frac{|bb^* + g|}{|g|} d\lambda \leq \kappa \qquad \text{(processing)}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda s} b \, d\lambda = 0, \quad s < -\tau \qquad \text{(availability)}$$

Main result

Theorem 1. *The optimal pair* (b, g) *is given by*

$$b = \frac{1}{\theta} g(a - \psi) \quad \textit{and} \quad g = \theta U \,\mathrm{diag}\left(\max\left\{1 - \frac{\theta}{d_i}, 0\right\}\right) U^*$$

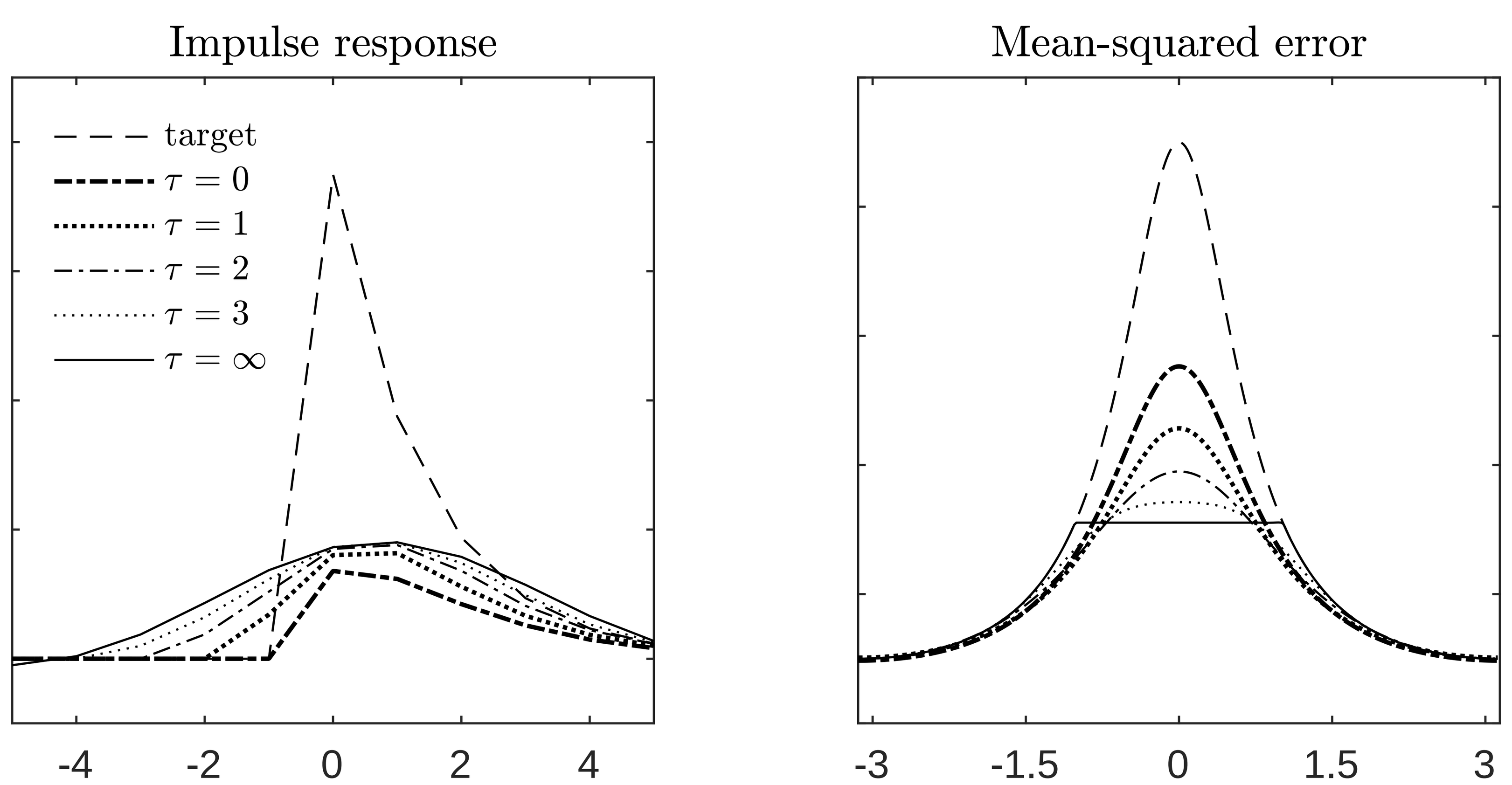
where $(a - \psi)(a - \psi)^* = U \,\mathrm{diag}(d_i) U^*$, *and* θ *and* ψ *solve*

$$\theta = \exp\left(-\frac{2\kappa}{n_x} + \frac{1}{2\pi n_x} \int_{-\pi}^{\pi} \sum_{i=1}^{n_x} \ln \max\{d_i, \theta\} d\lambda\right) \qquad (1)$$

$$\psi = \left[a - \theta U \,\mathrm{diag}\left(\frac{1}{\max\{d_i, \theta\}}\right) U^*(a - \psi)\right]_{-\tau} \qquad (2)$$

Forward-looking behavior

RI produces forward-looking behavior even with a backward-looking target



- Look at the $\tau = \infty$ case (frequency-domain approach makes this possible)
- Agent is trying to ignore the least important frequencies
- The "cost" of doing this is that the agent must be inattentive to the timing of the disturbances (proof of this **uncertainty principle** is in the paper)
- The agent cares less about timing than frequencies, even when $\tau < \infty$

Numerical algorithm

Initialize θ and ψ on a grid over $[-\pi, \pi]$, then iterate on (1) and (2)

- Use Matlab's `ifft` to evaluate the integral in (1)
- Use Matlab's `ifft` and `fft` to evaluate the operator $[\,\cdot\,]_{-\tau}$ in (2)

Advantages	Disadvantages
No state-space structure needed	Requires stationary target
No "curse" in state dimension	Slower for smaller states
Can handle first-best case when $\tau = \infty$	

- The paper compares this algorithm with a time-domain algorithm in the context of two examples with closed-form solutions: VAR(1) and MA(q)

Equilibrium model

Supplier of good i sets price according to

$$p_{it} = E_{it}[(1 - \xi)p_t + \xi q_t]$$

where $p_t \equiv \int p_{it} di$ and $q_t = \sum_{s=0}^{\infty} \delta_s \varepsilon_{t-s}$ is nominal expenditure.

- Target $x_t \equiv (1 - \xi)p_t + \xi q_t$ is *endogenous* when $\xi \neq 1$
- Model solved completely in the frequency domain using a nested loop
- When $\tau > 0$, expansionary nominal stimulus has *contractionary* effects