Erasmus
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The Block-Autoregressive Model in Non-Standard Bases

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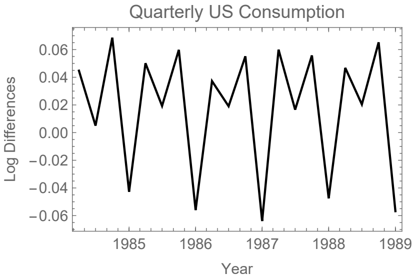
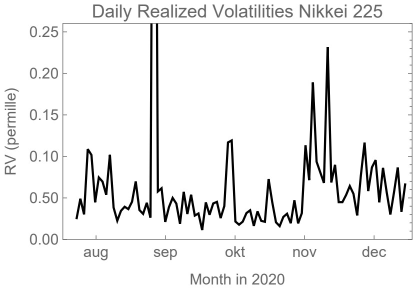
1 Examples of persistence





Persistence in time-series might involve

- seasonality, and / or
- long-memory dynamics.

The BAR model provides a unifying framework for both.

Data retrieved from the Federal Reserve Bank of St Louis and Oxford-Man Institute of Quantitative Finance.

1 Related literature

- Time series decomposition with different persistence levels
  - Bandi and Perron (2008)
  - Ortu et al. (2013)
  - Bandi et al. (2019a)
  - Bandi et al. (2019b)
  - Ortu et al. (2020)
- Periodic autoregressive (PAR) models
  - Osborn et al. (1988)
  - Birchenhall et al. (1989)
  - Franses (1994)

  With some recent applications by
  - Aknouche (2017)
  - Battaglia et al.(2020)
  - Baragona et al.(2021)

2 Unifying framework

The block-autoregressive model in the standard basis

$$\vec{y}_t^{\,s} = \mu + A\,\vec{y}_{t-s}^{\,s} + \,\vec{\varepsilon}_t^{\,s}$$

where

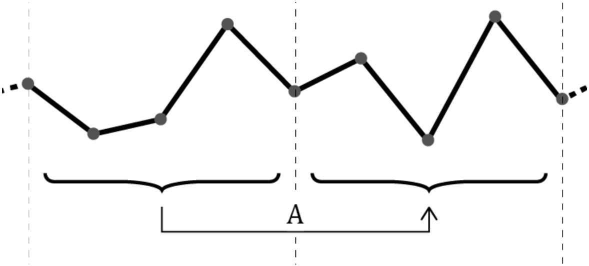
- s is the period
- $\vec{y}_t^s = \begin{bmatrix} y_t & y_{t-1} & \cdots & y_{t-s+1} \end{bmatrix}'$
- $\mu = \begin{bmatrix} \mu_s & \mu_{s-1} & \cdots & \mu_1 \end{bmatrix}'$
- $\vec{\varepsilon}_t^{\,s} \sim IID(0, \Sigma))$ with Σ positive definite
- A is an $s \times s$ matrix of AR coefficients

2 Example

Below an example model for $s = 4$.

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix} = \begin{bmatrix} \mu_4 \\ \mu_3 \\ \mu_2 \\ \mu_1 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} y_{t-4} \\ y_{t-5} \\ y_{t-6} \\ y_{t-7} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \varepsilon_{t-3} \end{bmatrix}$$

Observations are modeled in 'blocks' of s observations at a time.



2 Alternative representations using a basis change

Let Θ be an orthonormal matrix. We can write the BAR-model in a **time-domain** representation

$$\vec{y}_t^{\,s} = \mu + A\,\vec{y}_{t-s}^{\,s} + \vec{\varepsilon}_t^{\,s} \qquad \vec{\varepsilon}_t^{\,s} \sim IID(0,\Sigma))$$

and in a **scale-domain** representation

$$\begin{aligned} \Theta\,\vec{y}_t^{\,s} &= \Theta\mu + \Theta A \vec{y}_{t-s}^{\,s} + \Theta\vec{\varepsilon}_t^{\,s} \\ &= \Theta\mu + \Theta A \Theta' \Theta\,\vec{y}_{t-s}^{\,s} + \Theta\,\vec{\varepsilon}_t^{\,s} \\ \tilde{y}_t^{(s)} &= \tilde{\mu} + \tilde{A}\,\tilde{y}_{t-s}^{(s)} + \tilde{\varepsilon}_t^{(s)} \quad \tilde{\varepsilon}_t^{\,s} \sim IID(0,D)) \end{aligned}$$

where $D = \Theta\Sigma\Theta'$.

Changing the basis

1. yields parsimonious representations,
2. introduces a basis-indifferent decomposition procedure.

2 Simple example

We illustrate this with an example. Consider the autoregressive version of the scale-specific model (Bandi et. al. 2019). Let Θ be the Haar wavelet-basis and $s = 4$ using autocorrelation $\rho = 0.5$.

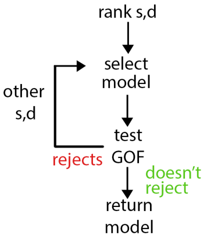
scale-domain
$$\begin{bmatrix} y_{t,1}^{(4)} \\ y_{t,2}^{(4)} \\ y_{t,3}^{(4)} \\ y_{t,4}^{(4)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-4,1}^{(4)} \\ y_{t-4,2}^{(4)} \\ y_{t-4,3}^{(4)} \\ y_{t-4,4}^{(4)} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t,1}^{(4)} \\ \varepsilon_{t,2}^{(4)} \\ \varepsilon_{t,3}^{(4)} \\ \varepsilon_{t,4}^{(4)} \end{bmatrix}$$

time-domain
$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix} = \begin{bmatrix} 0.12 & 0.12 & -0.18 & 0. \\ 0.12 & 0.12 & -0.18 & 0. \\ -0.18 & -0.18 & 0.25 & 0. \\ 0. & 0. & 0. & 0. \end{bmatrix} \begin{bmatrix} y_{t-4} \\ y_{t-5} \\ y_{t-6} \\ y_{t-7} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \varepsilon_{t-3} \end{bmatrix}$$

3 Estimation and inference

Model estimation involves

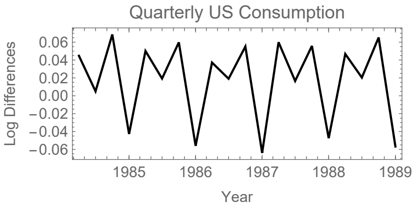
- estimation of A, Σ with least squares,
- estimation of Θ using an eigenvalue decomposition,
- model selection (s, d) based on the adj. R^2, and
- goodness of fit testing using a χ^2-statistic.



Basis Identifiability Assumption [BIA]

The scale domain residual covariance matrix $\tilde{\Sigma}$ is a diagonal matrix with diagonal elements $\sigma_1, \ldots, \sigma_s$ that are all unique and in descending order, i.e., $\sigma_1 > \ldots > \sigma_s$.

4 Empirical application



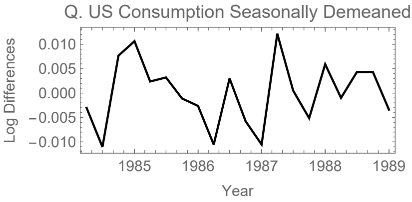


Table: Estimated parameters for log differences of US Consumption between January 1984 and January 2019.

$$\tilde{\mu} = \begin{bmatrix} 0.00 \\ -0.04 \\ 0.04 \\ 0.05 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \color{red}{0.4} & 0.3 & -0.4 & -0.3 \\ 0.2 & 0.1 & -0.4 & 0.4 \\ -0.2 & 0.3 & 0.1 & -0.1 \\ -0.1 & -0.2 & 0. & -0.1 \end{bmatrix} \quad diag(\tilde{\Sigma}) = \begin{bmatrix} \color{red}{0.9} \\ 0.6 \\ 0.2 \\ 0.1 \end{bmatrix} \quad \Theta = \begin{bmatrix} \color{red}{0.1} & \color{red}{0.9} & \color{red}{0.4} & \color{red}{0.2} \\ -0.9 & 0.1 & 0.1 & -0.3 \\ 0.3 & 0.1 & 0.1 & -0.9 \\ -0.02 & 0.4 & -0.9 & -0.1 \end{bmatrix}$$

The BAR model finds evidence for business cycle dynamics.

5 Conclusion

The BAR model

> realizes a unifying framework for time series with apparent or obfuscated persistence,

> realizes a basis indifferent time-series decomposition method, and

> is straight-forward to estimate, test, and apply.