Modeling and Decoupling Systemic Risk into Endopathic and Exopathic Competing Risks

Jingyu Ji[†‡], Deyuan Li[†], Zhengjun Zhang[‡]

[†]Fudan University, [‡]University of Wisconsin - Madison

Challenges in systemic risk management

- Systemic risk refers to the risk of collapse of an entire complex system due to the actions taken by the individual component entities or agents that comprise the system. Systemic risk may occur in almost every area, for example, financial crisis, flooding, forest fire, earthquake, market crash, economic crisis, global disease pandemic (like COVID-19), among many others (see [6, 7]). Typically, a system contains a number of risk sources, and once one comes first to collapse, the whole system is affected immediately, i.e., the risk sources are competing. When a disaster event (systemic risk) occurs, it may not be known what causes the event, i.e., its risk source. In such a scenario, it is of significance to decompose systemic risk into competing risks for learning risk patterns and better risk management.
- Internal risk refers to the risk from shocks that are generated and amplified within the system. It stands in contrast to external risk, which relates to shocks that arrive from outside the system.
- The occurrence of systemic risk is strongly correlated with extreme events. Modeling systemic risk through modeling extreme events is one of the essential topics in risk management.

New extreme value theory for maxima of maxima

Most recently, the accelerated max-stable distribution has been proposed by [1] to fit the extreme values of data generated from a mixture process (i.e., from different sources), whose mixture patterns vary with the time or sample size.

Suppose that the independent mixed sequence of random variables $\{X_i\}_{i=1}^n$ is composed of k subsequences $\{X_{j,i}\}_{i=1}^{n_j}$, $j=1,2,\ldots,k$; $\{X_{j,i}\}_{i=1}^{n_j} \overset{i.i.d.}{\sim} F_j(x)$, $n_j \to \infty$ as $n \to \infty$ and $n = n_1 + \cdots + n_k$. Denote $M_{j,n_j} = \max(X_{j,i}, i = 1,\ldots,n_j)$ as the maximum of the jth subsequence, $j = 1,2,\ldots,k$. Suppose $F_j \in MDA(H_j)$, where H_j is one of the three types of extreme value distritions, i.e., M_{j,n_j} has the following limit distribution with some norming constants $a_{j,n_j} > 0$ and centering constants b_{j,n_j},

$$\lim_{n\to\infty} P(a_{j,n_j}(M_{j,n_j} - b_{j,n_j}) \le x) = H_j(x). \tag{1}$$

Define $M_n = \max(M_{1,n_1}, M_{2,n_2},\ldots,M_{k,n_k})$, i.e., M_n is the maxima of k maxima of M_{j,n_j}s. For $k=2$, the limit distribution of M_n as $n \to \infty$ can be determined in the following cases:

Case 1. If $\frac{a_{1,n_1}}{a_{2,n_2}} \to a > 0$, $a_{1,n_1}(b_{2,n_2} - b_{1,n_1}) \to b < +\infty$, for some constants a and b, then

$$P(a_{2,n_2}(M_n - b_{2,n_2}) \le x) \to H_1(ax+b)H_2(x). \tag{2}$$

Case 2. If $\frac{a_{1,n_1}}{a_{2,n_2}} \to 0$, $a_{1,n_1}(b_{2,n_2} - b_{1,n_1}) \to +\infty$, then

$$P(a_{2,n_2}(M_n - b_{2,n_2}) \le x) \to H_2(x). \tag{3}$$

Autoregressive conditional accelerated Fréchet (AcAF)

Suppose Q_{kt}, $k = 1,\ldots,d$ are latent processes, and $Q_t = \max_{1\le k\le d} Q_{kt}$ where each $Q_{kt} = \max_{1\le i\le p_{kt}} X_{k,i,t}$ is again maxima of many time series at time t. Following [9] and [4], we assume

$$Q_{kt} = \mu_{kt} + \sigma_{kt} Y_{kt}^{1/\alpha_{kt}}, \tag{4}$$

where μ_{kt}, σ_{kt} and α_{kt} are the location, scale, and shape parameters with Y_{kt} being a unit Fréchet random variable with the distribution function $F(y) = e^{-1/y}$, $y > 0$.

Specifically, we consider two latent processes Q_{1t} and Q_{2t} to represent maximum negative log-returns across a group of stocks or of a particular stock's high-frequency trading whose price changes are driven by normal trading behavior and external information (e.g., sentiments), respectively.

The resulting maximum negative log-returns across that group of stocks or of that particular stock's high-frequency trading can be expressed as $Q_t = \max(Q_{1t}, Q_{2t}) = \max(\max_{1\le i\le p_{1t}} X_{1,i,t}, \max_{1\le i\le p_{2t}} X_{2,i,t})$, where each $\{X_{k,i,t}\}_{i=1}^{p_{kt}}$, $k=1,2$, is a set of time series whose price changes are due to corresponding price change driving factors, respectively.

For model parsimony, we assume $\mu_{1t} = \mu_{2t} = \mu_t$, $\sigma_{1t} = \sigma_{2t} = \sigma_t$, and follow the literature to assume μ_t as a constant and focus on the dynamics of σ_t, α_{1t} and α_{2t}, which are the pivotal parameters of modeling systemic risk and identifying risk sources. For the rest of the paper, we consider the following model:

$$Q_t = \max(Q_{1t}, Q_{2t}) = \mu + \sigma_t \max(Y_{1t}^{1/\alpha_{1t}}, Y_{2t}^{1/\alpha_{2t}}), \tag{5}$$

$$\log\sigma_t = \beta_0 + \beta_1 \log\sigma_{t-1} - \beta_2 \exp(-\beta_3 Q_{t-1}), \tag{6}$$

$$\log\alpha_{1t} = \gamma_0 + \gamma_1 \log\alpha_{1,(t-1)} + \gamma_2 \exp(-\gamma_3 Q_{t-1}), \tag{7}$$

$$\log\alpha_{2t} = \delta_0 + \delta_1 \log\alpha_{2,(t-1)} + \delta_2 \exp(-\delta_3 Q_{t-1}), \tag{8}$$

where $\{Y_{1t}\}$ and $\{Y_{2t}\}$ are sequences of independent and identically distributed (i.i.d.) unit Fréchet random variables. Assume $var(\gamma_2 \exp(-\gamma_3 Q_{t-1})) > var(\delta_2 \exp(-\delta_3 Q_{t-1}))$ for model identifiability.

Endopathic risk and exopathic risk: Definition

- When one of γ_2 and δ_2 is zero,
  - α_{1t} is called the tail index implied endopathic risk (for simplicity, call it endopathic risk),
  - α_{2t} is called the tail index implied exopathic risk (for simplicity, call it exopathic risk).
- When both γ_2 and δ_2 are zero, we define α_{1t} as the endopathic risk, while the exopathic risk is not defined.
- When both γ_2 and δ_2 are greater than zero, we refer α_{1t} to the endopathic risk and α_{2t} to the exopathic risk.
- When γ_1 and γ_3 (or δ_1 and δ_3) are zero, we define α_{1t} as the endopathic risk and α_{2t} as the exopathic risk.

Stationarity and ergodicity

For the AcAF model with $\beta_0, \gamma_0, \delta_0, \mu \in \mathbb{R}$, $\beta_2, \beta_3, \gamma_2, \gamma_3, \delta_2, \delta_3 > 0$, and $0 \le \beta_1 \ne \gamma_1 \ne \delta_1 < 1$, the latent process $\{\sigma_t, \alpha_{1t}, \alpha_{2t}\}$ is stationary and geometrically ergodic.

Maximum likelihood estimation

$\boldsymbol{\theta}_0 = (\beta_0^0, \beta_1^0, \beta_2^0, \beta_3^0, \gamma_0^0, \gamma_1^0, \gamma_2^0, \gamma_3^0, \delta_0^0, \delta_1^0, \delta_2^0, \delta_3^0, \mu_0)^T$.

The conditional probability density function (p.d.f.) of Q_t given $(\mu, \sigma_t, \alpha_{1t}, \alpha_{2t})^T$ is

$$f_t(\boldsymbol{\theta}) = \{\alpha_{1t}\sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}-1} + \alpha_{2t}\sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}-1}\} \times \exp\{-\sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}} - \sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}}\}. \tag{9}$$

By conditional independence, the log-likelihood function with observations $\{Q_t\}_{t=1}^n$ is

$$L_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{t=1}^n l_t(\boldsymbol{\theta}) = \frac{1}{n}\sum_{t=1}^n \left[\log\left\{\alpha_{1t}\sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}-1} + \alpha_{2t}\sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}-1}\right\} - \sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}} - \sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}}\right], \tag{10}$$

where $\{\sigma_t, \alpha_{1t}, \alpha_{2t}\}_{t=1}^n$ can be obtained recursively through (6)-(8), with an initial value $(\sigma_1, \alpha_{11}, \alpha_{21})^T$.

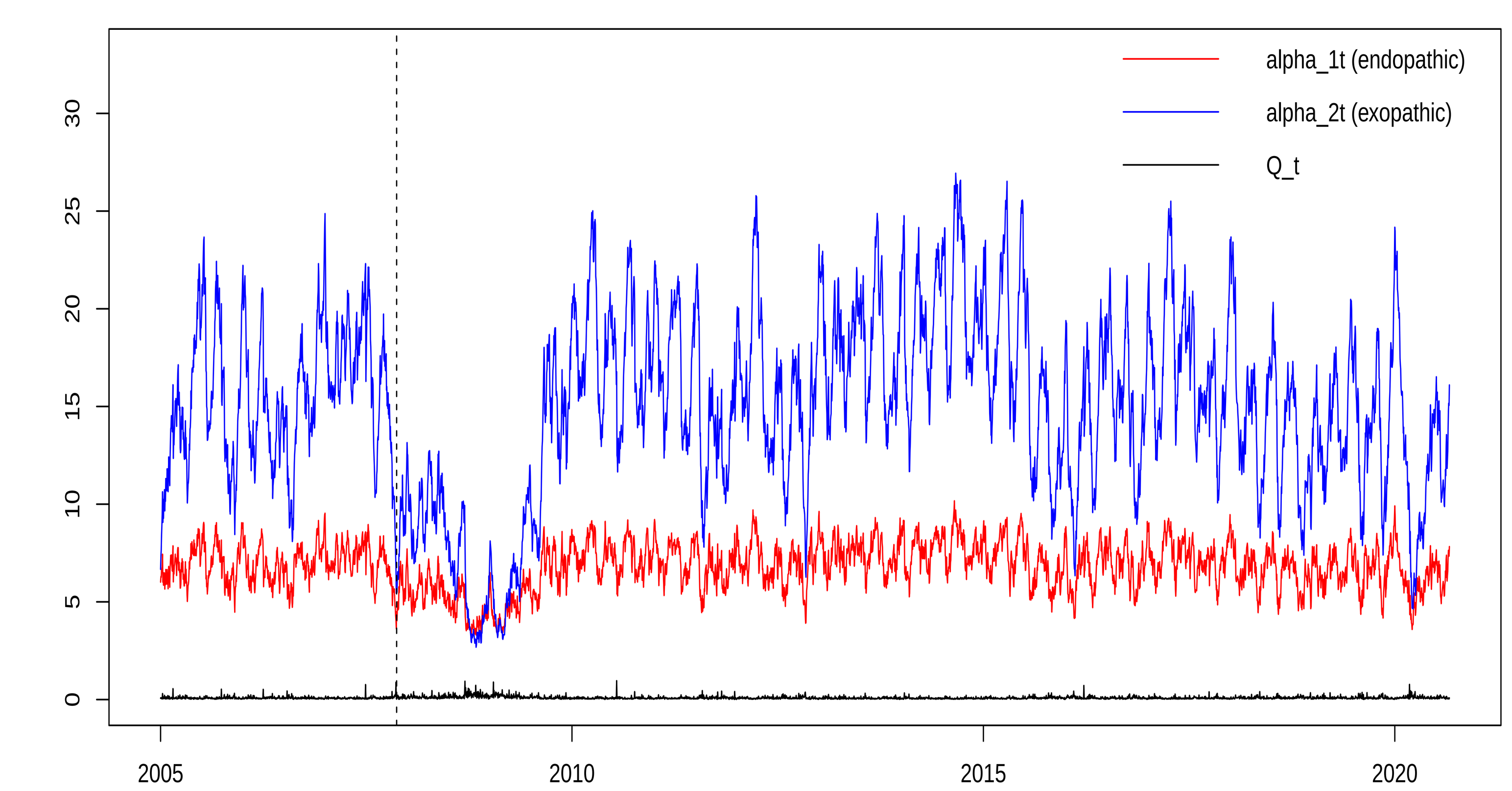
Theorem (Consistency)

Assume Θ is a compact set of the parameter space Θ_s. Suppose the observations $\{Q_t\}_{t=1}^n$ are generated by a stationary and ergodic model with true parameter $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_0$ is in the interior of Θ, then there exists a sequence $\hat{\boldsymbol{\theta}}_n$ of local maximizer of $\tilde{L}_n(\boldsymbol{\theta})$ such that $\hat{\boldsymbol{\theta}}_n \to_p \boldsymbol{\theta}_0$ and $||\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0|| \le \tau_n$, where $\tau_n = O_p(n^{-r})$, $0 < r < 1/2$. Hence $\hat{\boldsymbol{\theta}}_n$ is consistency.

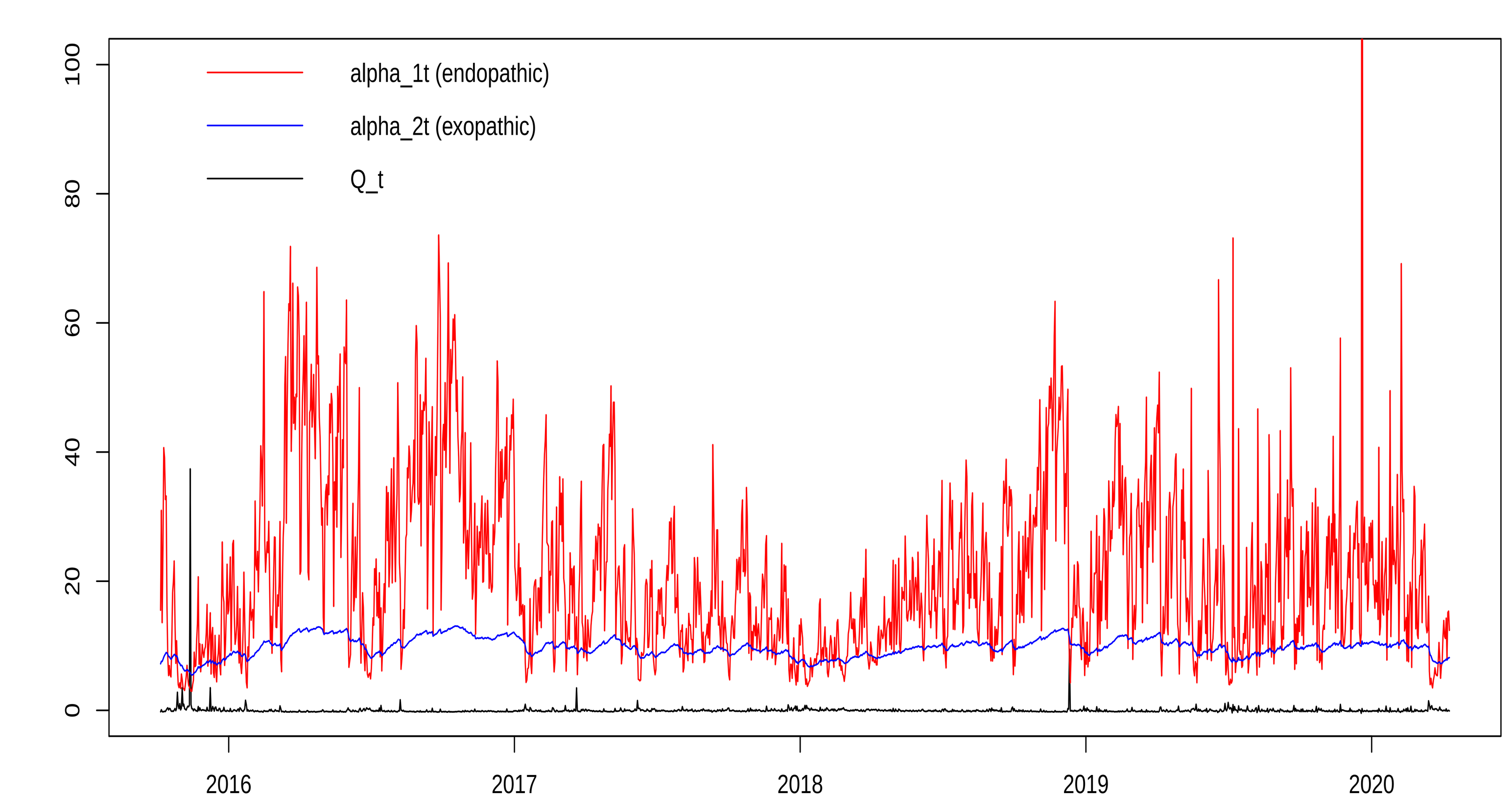
Theorem (Asymptotic normality)

Under the conditions in Theorem 1, we have $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} N(\mathbf{0}, \mathbf{M}_0^{-1})$, where $\hat{\boldsymbol{\theta}}_n$ is given in Theorem 1 and $\mathbf{M}_0$ is the Fisher Information matrix evaluated at $\boldsymbol{\theta}_0$. Further, the sample variance-covariance matrix of plug-in estimated score functions $\{\frac{\partial}{\partial\boldsymbol{\theta}} l_t(\hat{\boldsymbol{\theta}}_n)\}_{t=1}^n$ is a consistent estimator of $\mathbf{M}_0$.

Results



Estimated tail indices and cross-sectional maximum daily negative log-returns $\{Q_t\}$ (black) of S&P500 Index from January 3, 2005 to August 31, 2020.



Estimated tail indices and intra-day maxima of 5-minute negative log-returns $\{Q_t\}$ (black; normalized) from October 8, 2015 to April 9, 2020 for BTC/USD data.

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- Systemic risk may occur in almost every area, for example, financial crisis, flooding, forest fire, earthquake, market crash, economic crisis, global disease pandemic (like COVID-19), among many others (see [1, 2]).
- Typically, a system contains a number of risk sources, and once one comes first to collapse, the whole system is affected immediately, i.e., the risk sources are competing. When a disaster event (systemic risk) occurs, it may not be known what causes the event, i.e., its risk source. In such a scenario, it is of significance to decompose systemic risk into competing risks for learning risk patterns and better risk management.
- Internal risk refers to the risk from shocks that are generated and amplified within the system. It stands in contrast to external risk, which relates to shocks that arrive from outside the system.
- The occurrence of systemic risk is strongly correlated with extreme events. Modeling systemic risk through modeling extreme events is one of the essential topics in risk management.

Contributions of the new work

1. First and foremost, we propose a new decoupling risk framework to handle systemic risk. We decouple the systemic risk into endopathic risk and exopathic risk, which is the first based on our knowledge in the field.
2. Second, the empirical analysis shows our model's superior performance in two financial markets: the U.S. stock market and the Bitcoin trading market.
3. For the U.S. stock market, we find that exopathic risks are more volatile than endopathic risks. Under normal market conditions, endopathic risks dominate the stock market price fluctuations, while under turbulent market conditions, exopathic risks dominate.
4. For the Bitcoin trading market, endopathic risks are more volatile than exopathic risks. Exopathic risks dominate the cryptocurrency market price fluctuations under normal market conditions, while under turbulent market conditions, endopathic risks dominate.

References

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Classical extreme value theory

According to the Fisher-Tippett-Gnedenko Theorem ([2]; [3]), under certain conditions, the sample maxima can be accurately approximately modeled by a GEV distribution

$$G_{\mu,\sigma,\xi}(x) = \begin{cases} \exp\left\{ -\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}\right\}, \xi \neq 0, \\ \exp\left\{ -\exp(-\frac{x-\mu}{\sigma})\right\}, \qquad \xi = 0 \end{cases}$$

with $1+\xi(x-\mu)/\sigma > 0$, the location parameter $\mu \in \mathbb{R}$, scale parameter $\sigma > 0$, and shape/tail parameter $\xi \in \mathbb{R}$.

New extreme value theory

Most recently, the accelerated max-stable distribution has been proposed by [1] to fit the extreme values of data generated from a mixture process (i.e., from different sources), whose mixture patterns vary with the time or sample size.

Suppose that the independent mixed sequence of random variables $\{X_i\}_{i=1}^n$ is composed of k subsequences $\{X_{j,i}\}_{i=1}^{n_j}$, $j = 1, 2, \ldots, k$; $\{X_{j,i}\}_{i=1}^{n_j} \overset{i.i.d.}{\sim} F_j(x)$, $n_j \to \infty$ as $n \to \infty$ and $n = n_1 + \cdots + n_k$.

Maxima of maxima

Denote $M_{j,n_j} = \max(X_{j,i}, i = 1, \ldots, n_j)$ as the maximum of the jth subsequence, $j = 1, 2, \ldots, k$. Suppose $F_j \in MDA(H_j)$, where H_j is one of the three types of extreme value distritions, i.e., M_{j,n_j} has the following limit distribution with some norming constants $a_{j,n_j} > 0$ and centering constants b_{j,n_j},

$$\lim_{n\to\infty} P(a_{j,n_j}(M_{j,n_j} - b_{j,n_j}) \le x) = H_j(x). \tag{1}$$

Define $M_n = \max(M_{1,n_1}, M_{2,n_2}, \ldots, M_{k,n_k})$, i.e., M_n is the maxima of k maxima of M_{j,n_j}s. For $k = 2$, the limit distribution of M_n as $n \to \infty$ can be determined in the following cases:

Case 1. If $\frac{a_{1,n_1}}{a_{2,n_2}} \to a > 0$, $a_{1,n_1}(b_{2,n_2} - b_{1,n_1}) \to b < +\infty$, for some constants a and b, then

$$P(a_{2,n_2}(M_n - b_{2,n_2}) \le x) \to H_1(ax+b)H_2(x). \tag{2}$$

Case 2. If $\frac{a_{1,n_1}}{a_{2,n_2}} \to 0$, $a_{1,n_1}(b_{2,n_2} - b_{1,n_1}) \to +\infty$, then

$$P(a_{2,n_2}(M_n - b_{2,n_2}) \le x) \to H_2(x). \tag{3}$$

References

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[1] Cao, W. and Zhang, Z. (2020). *Statistical Theory and Related Fields*, 1-21.

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Autoregressive conditional accelerated Fréchet

- Q_{kt}, $k = 1, ..., d$ are latent processes with $Q_{kt} = \max_{1 \le i \le p_{kt}} X_{k,i,t}$.
- $Q_t = \max_{1 \le k \le d} Q_{kt}$
- Following [1] and [2], we assume
$$Q_{kt} = \mu_{kt} + \sigma_{kt} Y_{kt}^{1/\alpha_{kt}}, \tag{1}$$
where μ_{kt}, σ_{kt} and α_{kt} are the location, scale, and shape parameters with Y_{kt} being a unit Fréchet random variable with the distribution function $F(y) = e^{-1/y},\ y > 0$.
- Consider $d = 2$, and Q_{1t} and Q_{2t} to represent maximum negative log-returns across a group of stocks or of a particular stock's high-frequency trading whose price changes are driven by normal trading behavior and external information (e.g., sentiments), respectively.

For the rest of the presentation, we consider the following model:

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$$\log \sigma_t = \beta_0 + \beta_1 \log \sigma_{t-1} - \beta_2 \exp(-\beta_3 Q_{t-1}), \tag{3}$$
$$\log \alpha_{1t} = \gamma_0 + \gamma_1 \log \alpha_{1,(t-1)} + \gamma_2 \exp(-\gamma_3 Q_{t-1}), \tag{4}$$
$$\log \alpha_{2t} = \delta_0 + \delta_1 \log \alpha_{2,(t-1)} + \delta_2 \exp(-\delta_3 Q_{t-1}), \tag{5}$$

where $\{Y_{1t}\}$ and $\{Y_{2t}\}$ are sequences of independent and identically distributed (i.i.d.) unit Fréchet random variables. Assume $var(\gamma_2 \exp(-\gamma_3 Q_{t-1})) > var(\delta_2 \exp(-\delta_3 Q_{t-1}))$ for model identifiability.

Endopathic risk and exopathic risk: Definition

- When one of γ_2 and δ_2 is zero,
  - α_{1t} is called the tail index implied endopathic risk (for simplicity, call it endopathic risk),
  - α_{2t} is called the tail index implied exopathic risk (for simplicity, call it exopathic risk).
- When both γ_2 and δ_2 are zero, we define α_{1t} as the endopathic risk, while the exopathic risk is not defined.
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- When γ_1 and γ_3 (or δ_1 and δ_3) are zero, we define α_{1t} as the endopathic risk and α_{2t} as the exopathic risk.

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Stationarity and ergodicity

For the AcAF model with $\beta_0, \gamma_0, \delta_0, \mu \in \mathbb{R}$, $\beta_2, \beta_3, \gamma_2, \gamma_3, \delta_2, \delta_3 > 0$, and $0 \leq \beta_1 \neq \gamma_1 \neq \delta_1 < 1$, the latent process $\{\sigma_t, \alpha_{1t}, \alpha_{2t}\}$ is stationary and geometrically ergodic.

Maximum likelihood estimation

$\boldsymbol{\theta}_0 = (\beta_0^0, \beta_1^0, \beta_2^0, \beta_3^0, \gamma_0^0, \gamma_1^0, \gamma_2^0, \gamma_3^0, \delta_0^0, \delta_1^0, \delta_2^0, \delta_3^0, \mu_0)^T$.

The conditional probability density function (p.d.f.) of Q_t given $(\mu, \sigma_t, \alpha_{1t}, \alpha_{2t})^T$ is

$$\begin{aligned} f_t(\boldsymbol{\theta}) = &\{\alpha_{1t}\sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}-1} + \alpha_{2t}\sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}-1}\} \\ &\times \exp\{-\sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}} - \sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}}\}. \end{aligned} \tag{1}$$

By conditional independence, the log-likelihood function with observations $\{Q_t\}_{t=1}^n$ is

$$\begin{aligned} L_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{t=1}^n l_t(\boldsymbol{\theta}) = \frac{1}{n}\sum_{t=1}^n \Bigg[&\log\left\{\alpha_{1t}\sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}-1} + \alpha_{2t}\sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}-1}\right\} \\ &- \sigma_t^{\alpha_{1t}}(Q_t-\mu)^{-\alpha_{1t}} - \sigma_t^{\alpha_{2t}}(Q_t-\mu)^{-\alpha_{2t}}\Bigg], \end{aligned} \tag{2}$$

where $\{\sigma_t, \alpha_{1t}, \alpha_{2t}\}_{t=1}^n$ can be obtained recursively, with an initial value $(\sigma_1, \alpha_{11}, \alpha_{21})^T$.

Theorem (Consistency)

Assume Θ is a compact set of the parameter space Θ_s. Suppose the observations $\{Q_t\}_{t=1}^n$ are generated by a stationary and ergodic model with true parameter $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_0$ is in the interior of Θ, then there exists a sequence $\hat{\boldsymbol{\theta}}_n$ of local maximizer of $\tilde{L}_n(\boldsymbol{\theta})$ such that $\hat{\boldsymbol{\theta}}_n \to_p \boldsymbol{\theta}_0$ and $||\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0|| \leq \tau_n$, where $\tau_n = O_p(n^{-r})$, $0 < r < 1/2$. Hence $\hat{\boldsymbol{\theta}}_n$ is consistency.

Theorem (Asymptotic normality)

Under the conditions in Theorem 1, we have $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} N(\mathbf{0}, \mathbf{M}_0^{-1})$, where $\hat{\boldsymbol{\theta}}_n$ is given in Theorem 1 and $\mathbf{M}_0$ is the Fisher Information matrix evaluated at $\boldsymbol{\theta}_0$. Further, the sample variance-covariance matrix of plug-in estimated score functions $\{\frac{\partial}{\partial \boldsymbol{\theta}} l_t(\hat{\boldsymbol{\theta}}_n)\}_{t=1}^n$ is a consistent estimator of $\mathbf{M}_0$.

Proposition (Asymptotic uniqueness)

Denote $V_n = \{\boldsymbol{\theta} \in \Theta | \mu \leq cQ_{n,1} + (1-c)\mu_0\}$ where $Q_{n,1} = \min_{1\leq t\leq n} Q_t$, under the conditions in Theorem 1, for any fixed $0 < c < 1$. There exists a sequence of $\hat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta}\in V_n} \tilde{L}_n(\boldsymbol{\theta})$ such that, $\hat{\boldsymbol{\theta}}_n \to_p \boldsymbol{\theta}_0$, $||\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0|| \leq \tau_n$ where $\tau_n = O_p(n^{-r})$ with $0 < r < 1/2$, and

$$P(\hat{\boldsymbol{\theta}}_n \textit{ is the unique global maximizer of } \tilde{L}_n(\boldsymbol{\theta}) \textit{ over } V_n) \to 1.$$

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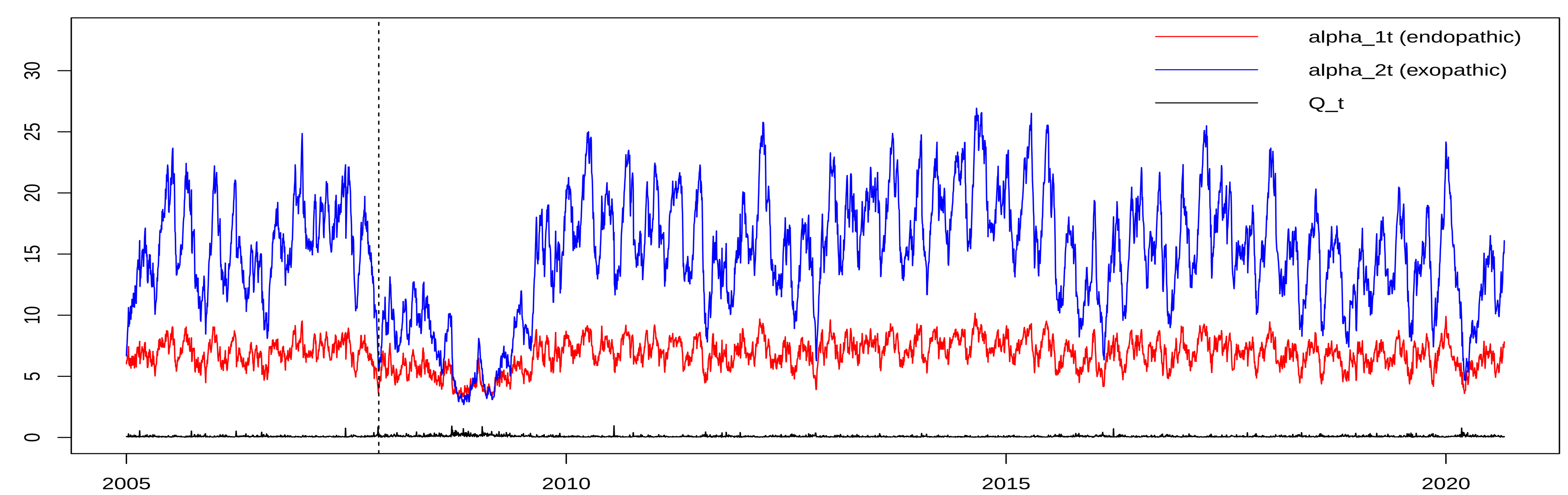
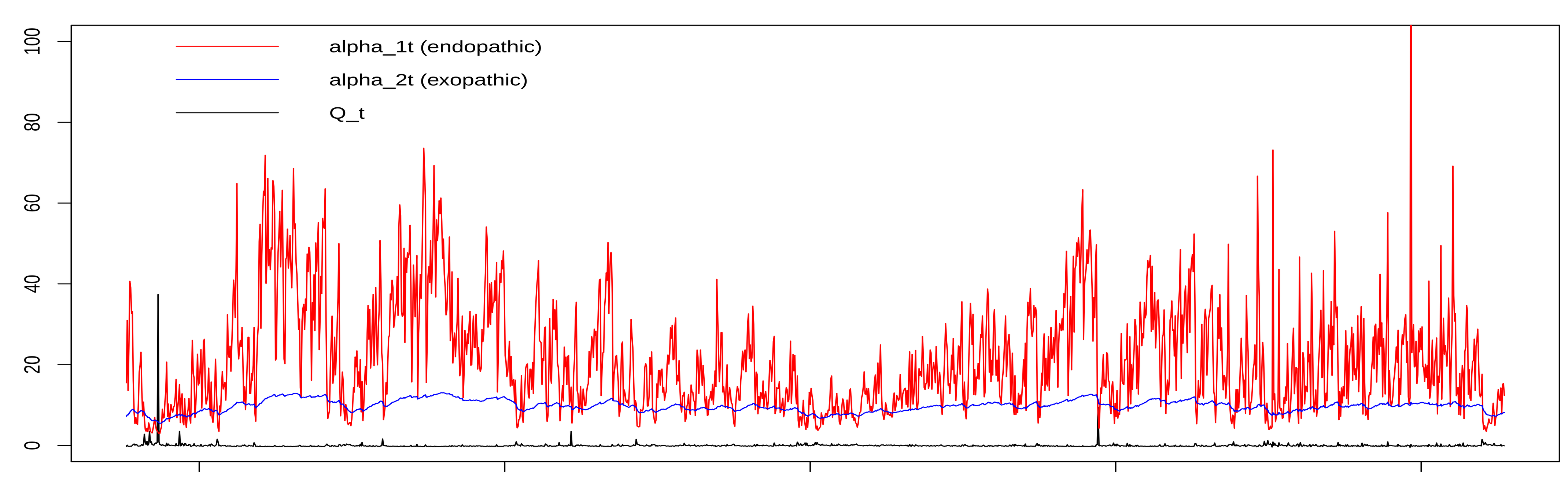
Results



Estimated tail indices and cross-sectional maximum daily negative log-returns $\{Q_t\}$ (black) of S&P500 Index from January 3, 2005 to August 31, 2020.



Estimated tail indices and intra-day maxima of 5-minute negative log-returns $\{Q_t\}$ (black; normalized) from October 8, 2015 to April 9, 2020 for BTC/USD data.

Discussions

- This paper develops a new autoregressive conditional accelerated Fréchet (AcAF) model for decoupling systemic financial risk into endopathic and exopathic competing risks.
- The AcAF model can be extended to many other aspects. One potential extension is to assume a dynamic structure for the location parameter μ. Another future direction is to extend two risk sources to multiple sources of risk with the construction of a flexible multivariate dynamic tail risk model.
- The AcAF model can be applied to diversified areas as long as decoupling systemic risks into competing endopathic risks and exopathic risks is concerned. These areas include systemic risks in social, political, economic, financial, market, regional, global, environmental, transportation, epidemiological, material, chemical, and physical systems.