Simultaneous Inference of a Partially Linear Model in Time Series

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Contributions

- Simultaneous inference of the non-parametric part of a partially linear model in “time series" is conducted when the non-parametric component is a “multivariate" unknown function.
- The developed methodology is applied to two examples in time series: (1) the forward premium regression and (2) a factor asset pricing model.

Model

- Partially linear time series model:

$$Y_i = Z_i^\top \boldsymbol{\beta} + \mu(X_i) + \epsilon_i, \quad i = 1, \cdots, T,$$

where ϵ_i is a random error with some general dependence structure. The goal of this study is to perform *specification test* for the null $H_0 : \mu(\cdot) = \mu_\theta(\cdot)$ for some $\theta \in \Theta$. Here $\mu_\theta(\cdot)$ is some parametric function with unknown θ.

Assumption 1

- Let the error process be:

$$\epsilon_i = \sum_{k=0}^{\infty} a_k \zeta_{i-k},$$

where ζ_i is an IID process. We require an algebraic decay rate of temporal dependence: For $\gamma > 0$ and $c_s > 0$, let

$$\sum_{k \geq i} |a_k| \leq c_s i^{-\gamma}, \; i \geq 1, \; \gamma > 0, \; c_s > 0$$

Assumption 2

- The kernel function $K(\cdot)$ is defined on $\mathbb{I} = [-1, 1]^d$ and is continuously differentiable up to order two.
- Assume $\max_{x \in \mathbb{I}} |K(x)| < \infty$ and $\int_{\mathbb{I}} K(x)dx = 1$. Also, assume $K(x)$ has its first-order derivative with $\sup_{x \in \mathbb{I}} \max_{1 \leq i \leq d} |\partial_i K(x)| < \infty$.
- Assume the bandwidth parameter $h \to 0$ and $h^d n \to \infty$.

Assumption 3

- Let $g(x|\mathcal{F}_{i-1})$ and X_{ij} have a finite q-th moment, for $q > 2$. Define

$$\begin{aligned}\theta_{k,q} &= \max_{x\in\mathbb{R}^d} \|g(x|\mathcal{F}_i) - g(x|\mathcal{F}_{i,\{i-k\}})\|_q \\ &\qquad + \|\max_{1\le j\le d} |X_{ij} - X_{ij,\{i-k\}}|\|_q.\end{aligned}$$

- Let $\sup_{m\geq 0} m^{\alpha'} \sum_{k\geq m} \theta_{k,q} < \infty$ for some $\alpha' > 0$.

Assumption 4

- Let Ω be some compact region. For some constants $c_g, c_g' > 0$, assume

$$c_g \leq \inf_{x \in \Omega} g(x) \leq \sup_{x \in \Omega} g(x) \leq c_g'.$$

  Additionally, assume $\sup_{x \in \Omega} \max_{1 \leq j \leq d} |\partial g(x) / \partial x_j| < \infty$.

- Assume, for any $s, t \in \Omega$, $\sup_{i_1, i_2} |g_{i_1, i_2}(s, t | \mathcal{F}_{i-1})| \leq c$ for some constant c. In addition, assume

$$\sup_{i_1, i_2} \left(\max_{1 \leq j \leq d} \left| \frac{\partial g_{i_1, i_2}(s, t \mid \mathcal{F}_{i-1})}{\partial s_j} \right| + \max_{1 \leq j \leq d} \left| \frac{\partial g_{i_1, i_2}(s, t \mid \mathcal{F}_{i-1})}{\partial t_j} \right| \right) \leq c',$$

for some constant $c' > 0$.

Assumption 5

- Assume that there exists some bounded function $h(\cdot)$: $\mathbb{R}^d \to \mathbb{R}^\ell$, such that

$$Z_i = h(X_i) + u_i,$$

where $h(\cdot)$ is Lipschitz-continuous and u_i are vectors in $\mathbb{R}^\ell$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (u_i u_i^\top) = B,$$

where B is a positive definite matrix.

- In addition, let

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n} \log n} \max_{1\le m\le n} |\sum_{k=1}^{m} u_{j_k}|_2 < \infty,$$

for any permutation $j_1, ..., j_n$ of the integers $1, 2, ..., n$. Moreover,

$$\max_{1\le i\le n} |u_i|_2 \le C.$$

Simultaneous Confidence Region

- Estimation of $\mu(\cdot)$ is achieved by:

$$\hat{\mu}_R(s) = \underset{\theta}{\operatorname{argmin}} \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{s - X_i}{h}\right) \left(Y_i - Z_i^\top \hat{\boldsymbol{\beta}}_R - \theta\right)^2,$$

where $\hat{\boldsymbol{\beta}}_R$ is the Robinson estimate (Robinson, 1988) of $\boldsymbol{\beta}$. This leads to:

$$\hat{\mu}_R(s) = \frac{1}{nh^d \hat{g}(s)} \sum_{i=1}^{n} K\left(\frac{s - X_i}{h}\right) \left(Y_i - Z_i^\top \hat{\boldsymbol{\beta}}_R\right),$$

with

$$\hat{g}(s) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{s - X_i}{h}\right).$$

Simultaneous Confidence Region

- Let S^2 be the the long-run variance of ϵ_i and $w_j(x_i)$ be the kernel weight function defined by:

$$w_j(x_i) = \frac{K\left(\frac{x_i - X_j}{h}\right)}{\sum_{k=1}^{n} K\left(\frac{x_i - X_k}{h}\right)}$$

- Denote $G_{i,\cdot} = (G_{i,1}, ..., G_{i,n})$, where $G_{i,j}$ is defined by $G_{i,j} = w_j(x_i) \cdot S$.

Asymptotic results

- Let $\eta \in \mathbb{R}^n$ be a standard normal random vector. If we consider $\delta = n^{-(d+1)/d}$, then $N = O(1/\delta^d) = O(n^{d+1})$. Under Assumptions 1–4, if $\log(n)^{1/2} h^{d+2} n \to 0$ and $(d+1)/q - \gamma + \log_n \log(n)^{1/2} < 0$, we have:

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}\left(\sup_{x \in \Omega} \left| \hat{\mu}(x) - \mu(x) \right| < u \right) - \mathbb{P}\left(\max_{1 \le i \le N} \left| G_{i,\cdot} \eta \right| < u \right) \right| \lesssim \Delta,$$

where $\hat{\mu}(x)$ is an “infeasible" estimate based on the true β and

$$\begin{aligned} \Delta &= (h^d n)^{-1/6} (\log Nn)^{7/6} + (n^{2/q}/(h^d n))^{1/3} \\ &\qquad + \log(Nn)^q (h^d n)^{-q/2+1} + C_n \end{aligned}$$

Asymptotic results

- Under the assumptions of Assumptions 1–5, if $\log(n)^{1/2} h^{d+2} n \to 0$ and $(d+1)/q - \gamma + \log_n \log(n)^{1/2} < 0$, we have:

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}\left(\sup_{x \in \Omega} \left| \hat{\mu}_R(x) - \mu(x) \right| < u \right) - \mathbb{P}\left(\max_{1 \leq i \leq N} \left| G_{i,.}\eta \right| < u \right) \right| \lesssim \Delta,$$

where $\hat{\mu}_R(x)$ is $\hat{\mu}(x)$ with the Robinson estimate (1988) in it.

Simultaneous Confidence Region

- Then, the p-th percentile *simultaneous confidence interval* for $\mu(\cdot)$ can be shown by:

$$\hat{\mu}_R(x) - z_p \leq \mu(x) \leq \hat{\mu}_R(x) + z_p,$$

where z_p is the p-th quantile for the $\max_i |G_{i,\cdot}\eta|$ and η is a standard Gaussian random vector.

Forward Premium Regression

- Consider the monetary model in Mark (1995):

$$s_{t+1} - s_t = \alpha + \beta(x_t - s_t) + u_{t+1}$$

where s_t is log of monthly *spot* exchange rate at time t and x_t is the equilibrium level of the spot exchange rate $x_t := m_t - m_t^* - \lambda(y_t - y_t^*)$ with m_t and y_t being log of domestic money stock and log of monthly production, respectively.

- We let $\lambda = 1$. One can rewrite the model as:

$$s_{t+1} - s_t = \alpha + \beta(x_t - f_{1,t}) + \beta(f_{1,t} - s_t) + u_{t+1}$$

where $f_{1,t}$ is log of monthly *forward* exchange rate with one-month maturity at time t.

Forward Premium Regression

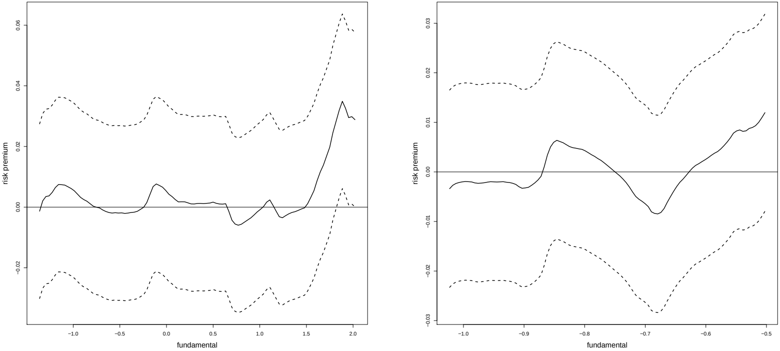
- A flexible way to model the risk premium is:

$$s_{t+1} - s_t = \mu(x_t - f_{1,t}) + \beta(f_{1,t} - s_t) + u_{t+1} \qquad (1)$$

where $\mu(\cdot)$ is some unknown function.

- The theory of Uncovered Interest Parity (*UIP*) implies $\mu(\cdot) = 0$, while numerous empirical studies actually show $\mu(\cdot) \neq 0$. Interestingly, (1) is a special case of the partially linear model framework. Hence the methodology developed here can be readily applied to decide whether or not the *UIP* condition holds.

Forward Premium Regression



(a) *GBP*/*USD* (b) *EUR*/*USD*

Conditional factor model

- Consider the single-factor asset pricing model:

$$R_{it} = \alpha(X_t) + \beta R_t^m + \zeta_{it} \tag{2}$$

where R_{it} is the excess return of momentum portfolio and R_t^m is the excess return on the value-weighted market index portfolio at t, respectively.

- Here $\alpha(\cdot)$ is the *pricing error* of the factor model that depends on X_t, a vector of random variables. The pricing error in (2) is likely to be time-varying and its variation is related to X_t. We let X_t contain the size factor or the book-to-market ratio, etc.

Conditional factor model

- To that end, we consider a *bivariate* pricing error:

$$R_{it} = \alpha(SMB_t,\ HML_t) + \beta R_t^m + \zeta_{it} \tag{3}$$

where SMB_t and HML_t represent the size and book-to-market factors, respectively.

- Given that (3) is a special case of the partially linear model, our methodology readily applies here. By constructing a *SCR* for the unknown $\alpha(\cdot,\cdot)$, one can conduct simultaneous inference for the zero-pricing-error hypothesis for the factor model in (3).

Conditional factor model

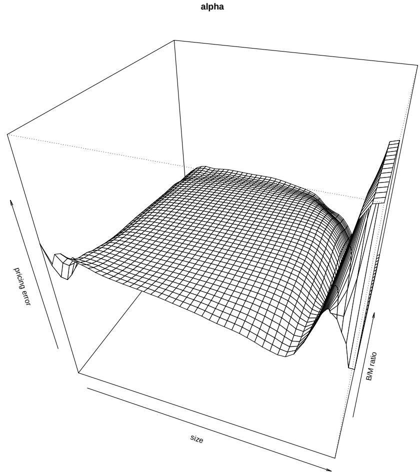
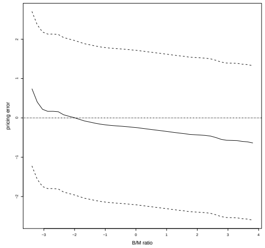
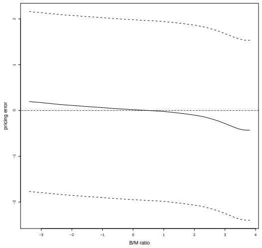


Figure:

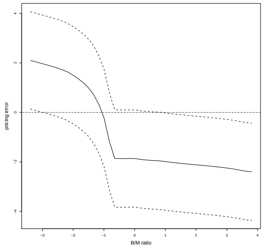
Conditional factor model



(a) 5th-percentile of size

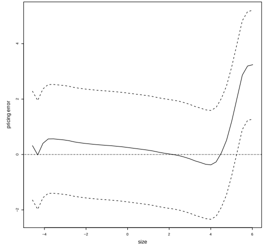


(b) 50th-percentile of size

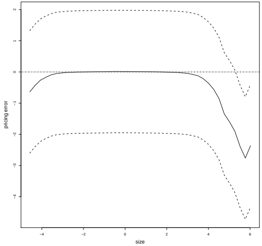


(c) 95th-percentile of size

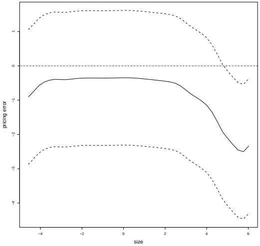
Conditional factor model



(a) 5th-percentile of B/M ratio



(b) 50th-percentile of B/M ratio



(c) 95th-percentile of B/M ratio

Summary

- We illustrate how to construct the simultaneous confidence region (*SCR*) for the multivariate unknown function in time series.

- The inference of the model is conducted through the construction of *SCR*, which is a multi-dimensional extension of the two-dimensional uniform confidence band.

- The zero-risk-premium hypothesis for *GBP*/*USD* is narrowly rejected at a 5 percent level, mainly due to the surge in the risk premium estimate when the fundamental takes on a large value.

- The hypothesis of zero-pricing-error is also rejected for the factor model, due to the underlying non-linear nature.