Inference of jumps using wavelet variance

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Objectives

Statistical inference of jumps in nonparametric regression models with long memory noise

- Propose **new** test statistic for the presence of jumps
- Propose **new** sequential applications of tests to estimate number of jumps and their locations

Roadmap

- Model
- Motivation
- Test statistic based wavelet variance
- Testing for hypothesis of no jumps
- Estimating number of jumps and their locations
- Simulation Study + Daily Dow-Jones Industrial Average

Model (1)

$$Y_i = f(i/n) + \varepsilon_i$$

- Under H_0 :

$$f(x) \equiv f_C(x),$$

where $f_C : [0,1] \longrightarrow \mathbf{R}$, f_C is continuously differentiable

- Under H_1 :

$$f(x) \equiv f_C(x) + f_J(x),$$

where $f_J(x) \equiv \sum_{l=1}^{m_0} d_l I\{x \geq \lambda_l\}$ m_0 is the finite number of jumps, λ_l's are jump locations, and d_l's are jump sizes

Model (2)

Recall

$$Y_i = f(i/n) + \varepsilon_i$$

- Autocorrelation of ε_i satisfies

$$corr\left(\varepsilon_i, \varepsilon_{i'}\right) \asymp \left|i - i'\right|^{-2+2H}, \; i - i' \to \infty,$$

  for $H \in [0.5, 1)$

- H represents Hurst parameter:
  - When $H = 0.5$, ε_i is an independent Gaussian error
  - When $H \in (0.5, 1)$, ε_i has long-range dependence (or long memory)

Motivation

- Model is very general to allow for nonparametric regression models with long memory noise
- Existing approaches from Wang (1995, 1999) based on the sup-type test statistic
- However, under H_0, Wang's sup-type test statistic converges very slowly to an extreme-value distribution
- **Solution**: our test statistic is based on robust estimator of the variance of the wavelet coefficients, and converges faster to normal distribution
- **Advantage** in finite sample:
    - Better size control for testing H_0
    - Better precision of estimating m_0 (number of jumps)

Wavelet Variance Estimation (1)

- Discrete wavelet transformation of $\mathbb{W}^{\mathbf{A}}_{j,k}$ at scale $j \in Z$ and location $k \in Z$:

$$\mathbb{W}^{\mathbf{A}}_{j,k} \equiv \frac{1}{n}\sum_{i=1}^{n} \psi_{j,k}\left(\frac{i}{n}\right) A_i,$$

  where

$$\psi_{j,k}\left(\frac{i}{n}\right) \equiv 2^{j/2}\psi\left(k - 2^j \frac{i}{n}\right),$$

  and $\psi(t)$ is a wavelet function, that is, $\int \psi(t)\, dt = 0$

- Then

$$\mathbb{W}^{\mathbf{Y}}_{j,k} = \mathbb{W}^{\mathbf{f}}_{j,k} + \mathbb{W}^{\mathbf{B}_H}_{j,k}$$

- Property of $\mathbb{W}^{\mathbf{f}}_{j,k}$ and $\mathbb{W}^{\mathbf{B}_H}_{j,k}$:
  - $\mathbb{W}^{\mathbf{f}}_{j,k}$ is spatially adaptive to pointwise smoothness of $f(x)$
  - $\mathbb{W}^{\mathbf{B}_H}_{j,k}$ decorrelates the long-memory noise ε_i

Wavelet Variance Estimation (2)

Recall

$$\mathbb{W}^{\mathbf{Y}}_{j,k} = \mathbb{W}^{\mathbf{f}}_{j,k} + \mathbb{W}^{\mathbf{B}_H}_{j,k}$$

- Under H_0 : for all k,

$$\mathbb{W}^{\mathbf{Y}}_{j,k} \approx \mathbb{W}^{\mathbf{B}_H}_{j,k},$$

  where $\mathbb{W}^{\mathbf{B}_H}_{j,k}$ is Gaussian
- Under H_1 : for locations k near jumps,

$$\mathbb{W}^{\mathbf{Y}}_{j,k} \approx \mathbb{W}^{\mathbf{f}}_{j,k} \approx \mathbb{W}^{\mathbf{f}_J}_{j,k};$$

  otherwise,

$$\mathbb{W}^{\mathbf{Y}}_{j,k} \approx \mathbb{W}^{\mathbf{B}_H}_{j,k},$$

  where $\mathbb{W}^{\mathbf{B}_H}_{j,k}$ is Gaussian
- **Remark**: Wang (1995, 1999) proposed the test statistic $\sup_{k\in\mathbf{K}} \left|\mathbb{W}^{\mathbf{Y}}_{j,k}\right|$ where $\mathbf{K} \equiv \left\{1, \cdots, 2^j\right\}$ for the jump detection

Wavelet Variance Estimation (3)

- Unlike Wang's approach, our test statistic is based on the second moment of wavelet coefficients
- We utilize the wavelet variance which measures the variability of wavelet coefficients
- Define wavelet variance at a given scale j by

$$\sigma_j^2 \equiv Var\left(\mathbb{W}_{j,1}^{\mathbf{B}_H}\right)$$

- Two different estimators of wavelet variance σ_j^2 :
  - Non-robust to $\mathbb{W}_{j,k}^{\mathbf{f}_J}$:

$$\widehat{\sigma}_{j,\mathbf{K}}^2 \equiv \frac{\sum_{k\in\mathbf{K}}(\mathbb{W}_{j,k}^{\mathbf{Y}})^2}{2^j};$$

  - Robust to $\mathbb{W}_{j,k}^{\mathbf{f}_J}$:

$$\widetilde{\sigma}_{j,\mathbf{K}}^2 \equiv \left[med_{k\in\mathbf{K}}\left|\frac{\mathbb{W}_{j,k}^{\mathbf{Y}}}{0.6745}\right|\right]^2.$$

Test Statistic for Hypothesis of No Jumps

Intuition: $\widetilde{\sigma}^2_{j,\mathbf{K}}$ is a robust estimator of σ^2_j regardless of the presence of jumps, and that $\widehat{\sigma}^2_{j,\mathbf{K}}$ is not robust to the presence of outliers

- (Infeasible) test statistic:

$$D_{j,\mathbf{K}} \equiv \frac{\widehat{\sigma}^2_{j,\mathbf{K}} - \widetilde{\sigma}^2_{j,\mathbf{K}}}{\sqrt{\omega}},$$

  where $\omega \equiv Var\left(\frac{\sum_{k=1}^{2^j}\left(\mathbb{W}^{\mathbf{B}_H}_{j,k}\right)^2}{2^j} - \left[med_{k\in\{1,\cdots,2^j\}}\left|\frac{\mathbb{W}^{\mathbf{B}_H}_{j,k}}{0.6745}\right|\right]^2\right)$

  (not depend on the presence of jumps)

- Under H_0 :

$$\lim_{n\to\infty} \Pr\left[|D_{j,\mathbf{K}}| \geq C_\gamma\right] = \gamma;$$

- Under H_1 :

$$\lim_{n\to\infty} \Pr\left[|D_{j,\mathbf{K}}| \geq C_\gamma\right] = 1,$$

  where $C_\gamma = \Phi^{-1}\left(1 - \frac{\gamma}{2}\right)$

Estimating Number of Jumps and Locations (1)

Sequential procedure based on $D_{j,\mathbf{K}}$:

Step 1 Conduct a test for $H_0 : m_0 = 0$ (no jump) against $H_1 : m_0 > 0$ (at least one jump). Reject H_0 if

$$\left|D_{j,\mathbf{K}}\right| > C_\gamma,$$

where $\mathbf{K} \equiv \left\{1, \cdots, 2^j\right\}$. If H_0 is not rejected, set $\widehat{m} = 0$;

Step 2 If $H_0 : m_0 = 0$ is rejected in Step 1, conduct a test for $H_0 : m_0 = 1$ (one jump) against $H_1 : m_0 > 1$ (at least two jumps). Reject H_0 if

$$\left|D_{j,\mathbf{K}\setminus\widehat{\mathbf{K}}_1}\right| > C_\gamma,$$

where

$$\widehat{\mathbf{K}}_1 \equiv \left\{k : \widehat{k}_1 - k \in \mathrm{supp}(\psi)\right\}$$

with $\widehat{k}_1 \equiv \arg\sup_{k\in\mathbf{K}} \left|\mathbb{W}^{\mathbf{Y}}_{j,k}\right|$. If H_0 is not rejected, set $\widehat{m} = 1$;

Estimating Number of Jumps and Locations (2)

Step 3 If $H_0 : m_0 = 1$ is rejected, conduct a test for $H_0 : m_0 = 2$ (two jumps) against $H_1 : m_0 > 2$ (at least three jumps). Reject H_0 if

$$\left| D_{j,\mathbf{K}\setminus\left(\widehat{\mathbf{K}}_1\cup\widehat{\mathbf{K}}_2\right)} \right| > C_\gamma,$$

where

$$\widehat{\mathbf{K}}_2 \equiv \left\{ k : \widehat{k}_2 - k \in \operatorname{supp}(\psi) \right\}$$

with $\widehat{k}_2 \equiv \arg\sup_{k\in\mathbf{K}\setminus\widehat{\mathbf{K}}_1} \left| \mathbb{W}^{\mathbf{Y}}_{j,k} \right|$. If H_0 is not rejected, set $\widehat{m} = 2$;

Step 4 Repeat the step until H_0 is not rejected, so that $\widehat{m}$ satisfies

$$\left| D_{j,\mathbf{K}\setminus\cup_{l=1}^{\widehat{m}}\widehat{\mathbf{K}}_l} \right| \leq C_\gamma,$$

where

$$\widehat{\mathbf{K}}_l \equiv \left\{ k : \widehat{k}_l - k \in \operatorname{supp}(\psi) \right\}$$

with $\widehat{k}_l \equiv \arg\sup_{k\in\mathbf{K}\setminus\cup_{l=1}^{l-1}\widehat{\mathbf{K}}_l} \left| \mathbb{W}^{\mathbf{Y}}_{j,k} \right|$ with $l = 1, \cdots, \widehat{m}$.

Estimating Number of Jumps and Locations (3)

Both estimated number of jumps and locations are consistent. Hence we have

$$\begin{aligned} \Pr\left(\widehat{m} = m_0\right) &\to 1, \\ \sum_{l=1}^{\widehat{m}_0} \left(\widehat{\lambda}_l - \lambda_l\right)^2 &= O_p\left(2^{-2j}\right), \end{aligned}$$

where

$$\widehat{\lambda}_l \equiv \frac{\widehat{k}_l}{2^j}.$$

Feasible Test Statistic

Recall

$$D_{j,\mathbf{K}} \equiv \frac{\widehat{\sigma}^2_{j,\mathbf{K}} - \widetilde{\sigma}^2_{j,\mathbf{K}}}{\sqrt{\omega}}$$

and

$$\omega \equiv Var\left(\frac{\sum_{k=1}^{2^j}\left(\mathbb{W}^{\mathbf{B}_H}_{j,k}\right)^2}{2^j} - \left[med_{k\in\{1,\cdots,2^j\}}\left|\frac{\mathbb{W}^{\mathbf{B}_H}_{j,k}}{0.6745}\right|\right]^2\right)$$

- Estimator $\widehat{\omega}$:
  - Rewrite the estimation errors of $\widehat{\sigma}^2_{j,\mathbf{K}}$ and $\widetilde{\sigma}^2_{j,\mathbf{K}}$ in terms of sample average
  - Truncate the $100 \times (1-\epsilon)$ percent of the largest $\left|\mathbb{W}^{\mathbf{Y}}_{j,k}\right|$ to construct the truncated version of sample averages
  - Apply Andrews (1991)'s long-run covariance estimation
- $\widehat{D}_{j,\mathbf{K}} \equiv \frac{\widehat{\sigma}^2_{j,\mathbf{K}} - \widetilde{\sigma}^2_{j,\mathbf{K}}}{\sqrt{\widehat{\omega}}}$

Simulation Study: See Paper

Daily Dow-Jones Industrial Average: See Paper