A bootstrap assisted second order stationarity test for nonlinear time series

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Motivation

While many tests of second order stationarity have been developed for linear or Gaussian time series, time series are often nonlinear and non-Gaussian in many econometrics and finance applications. A bootstrap assisted test is proposed to check the second order stationarity of nonlinear time series.

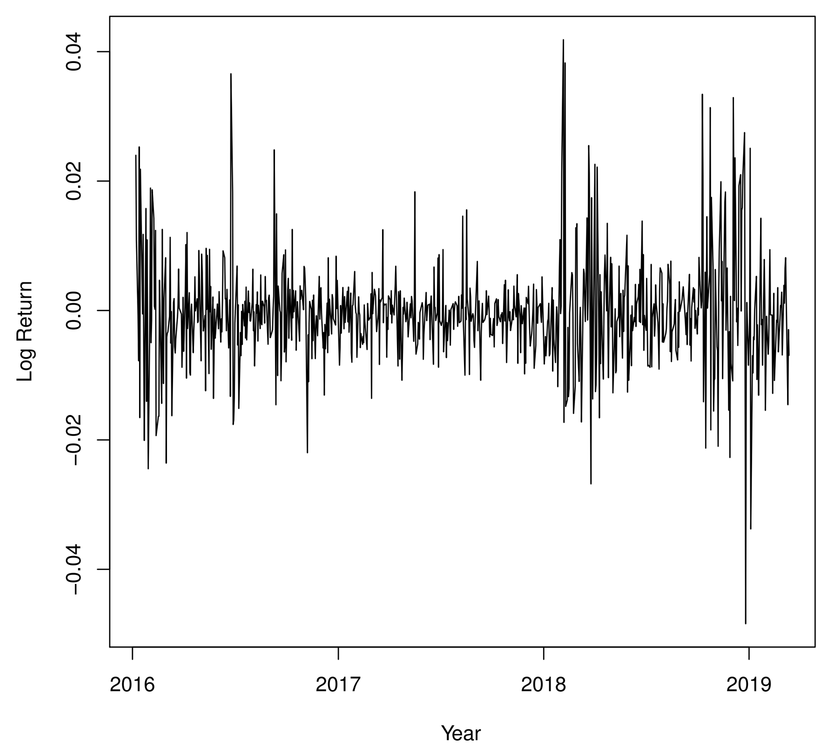

Fig. 1: SP 500 log return

Locally stationary nonlinear processes

Let $\{\epsilon_t\}_{t\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with mean zero and $\mathcal{F}_t = (\epsilon_t, \epsilon_{t-1}, \dots)$. A mean zero locally stationary process of [1] is

$$X_t = X_{t,T} = H_{t,T}(\mathcal{F}_t),$$

where $H_{t,T}$ is a measurable function, for each $t = 1, 2, \cdots, T$, and T is the length.

The local autocovariance function at lag h is

$$\gamma_h(u) = \mathrm{Cov}(Y_{t-h}(u), Y_t(u)) \text{ for } u \in [0,1],$$

where $\{Y_t(u)\}$ is the stationary approximation process for $\{X_{t,T}\}$ for each $u \in [0,1]$.

The null hypothesis:

$$H_0\colon \gamma_h(u) = c_h,$$

for all $u \in [0,1]$, $h = 0, 1, \dots$ a.s. A sequence of local alternatives:

$$H_{a,T}\colon \gamma_h(u) = c_h + l_T g_h(u),\ h = 0, 1, \dots,$$

where $l_T \to 0$ at some appropriate rate and $\int_0^1 g_h^2(u)du > 0$ at some lag h. If $l_T = 1$, it is a fixed alternative H_a.

Walsh transformation

The k-th Walsh ordinate at lag h:

$$\hat{d}_h^{(k)} = \frac{1}{T}\sum_{t=1}^{T-h} X_t X_{t+h} W_k\left(\frac{t-1}{T}\right),$$

where $W_k(x)$ is the k-th Walsh function in $x \in [0,1]$.

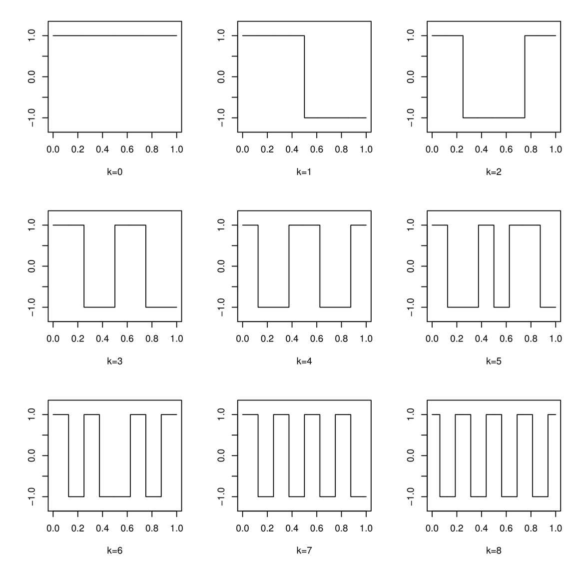

Fig. 2: Walsh functions

Theorem 1: Under some regularity assumptions and the stationarity, for integers $R = O(\log(T)/\log(\log(T)))$ and $K = O(T^{1/3})$, we have

$$\sup_{k=1,2,\dots,K,\,\mathbf{x}\in\Omega_R}\left|P(T^{1/2}\hat{\boldsymbol{d}}_R^{(k)} \le \mathbf{x}) - P\{(g_1,\dots,g_R)^T \le \mathbf{x}\}\right| \to 0,$$

as $T \to \infty$, where $\hat{\boldsymbol{d}}_R^{(k)} = (\hat{d}_0^{(k)}, \hat{d}_1^{(k)}, \dots, \hat{d}_{R-1}^{(k)})^{\mathrm{T}}$, Ω_R is the space of R dimensional vector of real numbers and $\{g_i\}_{i\ge1}$ is a mean zero Gaussian process with $\mathrm{cov}(g_i, g_j) = \Sigma_{i,j} = \lim_{T\to\infty} T\mathrm{Cov}(\hat{\gamma}_{i-1}, \hat{\gamma}_{j-1})$.

The Bootstrap covariance matrix estimate and the test statistic

A covariance matrix estimate $\hat{\boldsymbol{\Sigma}}_R$ is obtained via BOB bootstrap ([2]) with block size b to estimate $\boldsymbol{\Sigma}_r = \sqrt{T}(\hat{\gamma}_0, \hat{\gamma}_1, \dots, \hat{\gamma}_{r-1})^{\mathrm{T}}$.

Theorem 2: Under Assumptions in Theorem 1, and $b = o(T^{0.5})$, we have

$$|\hat{\boldsymbol{\Sigma}}_R - \boldsymbol{\Sigma}_R|_\infty \to 0$$

in probability, as $T \to \infty$, where $|\cdot|_\infty$ is the maximum norm defined to be the maximum absolute value among all elements of the matrix.

Following [3], the test statistic:

$$\hat{S}_{R,K} = \max_{1\le k\le K}\left[\max_{1\le r\le R}(\{\tilde{\boldsymbol{d}}_r^{(k)}\}^{\mathrm{T}}\{\tilde{\boldsymbol{d}}_r^{(k)}\} - 2r) - 2\sqrt{k-1}\right],$$

where $\tilde{\boldsymbol{d}}_R^{(k)} = \hat{\boldsymbol{P}}_R\hat{\boldsymbol{d}}_R^{(k_1)}$, and $\hat{\boldsymbol{P}}_R\hat{\boldsymbol{P}}_R^{\mathrm{T}} = \hat{\boldsymbol{\Sigma}}_R^{-1}$.

Asymptotic results

Theorem 3: Under Assumptions of Theorem 2,

$$\hat{S}_{R,K} \to \sup_{k\ge1}\left[\sup_{r\ge1}(\sum_{i=1}^r e_{k,i}^2 - 2r) - 2\sqrt{k-1}\right]$$

in distribution as $T \to \infty$, where $e_{k,i}, k, i = 1, 2, \dots,$ are mutually independent standard normal random variables.

Theorem 4: For integers $R = O(\log T/\log(\log(T)))$, $K = O(T^{1/3})$, $b = o(T^{0.5})$, and with some regularity conditions for locally stationary processes,

$$\hat{S}_{R,K} \to \infty$$

in probability as $T \to \infty$, under a fixed alternative or a sequence of local alternatives with $l_T T^{1/2} \to \infty$.

Simulation study

Nonlinear time series:

- Model S6: $X_t = \sigma_t Z_t$, where $\sigma_t^2 = 1.0 + 0.4X_{t-1}^2 + 0.3X_{t-2}^2$;
- Model S7: $X_t = 0.3X_{t-1} + 0.6X_{t-1}Z_{t-1} + Z_t$;
- Model S8: $X_t = \sigma_t Z_t$, where $\sigma_t^2 = 1.0 + 0.2X_{t-1}^2 + 0.4\sigma_{t-1}^2$.

$T = 256$, Rejection rates in percentage

Statistics	$\hat{S}_{R,K}, c_b = 3$		JWW of [3]		Nason's test in [4]	
α	0.1	0.05	0.1	0.05	0.1	0.05
S6	9.0	4.7	40.4	28.6	50.5	37.3
S7	5.5	3.6	30.4	23.9	56.3	42.4
S8	9.1	6.0	40.1	27.9	36.7	23.3

References

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